

Measure Concentration on Fermi Balls

Kurt Pagani

Dedicated to Maex on her 55th birthday

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Abstract

This note reports on some attempts to examine if and under which conditions the naturally scaled probability measures associated to an orthonormal basis of a classical Paley-Wiener space converge to a uniform distribution (on a compact set in momentum space). The results are still quite unsatisfactory, yet we got some indications that for inf-compact functions (symbols) and domains for which some generalized Weyl law holds, the measures converge weakly to a (generalized) Fermi ball.

1 Notation and Results

Let Ω be an open subset of \mathbb{R}^n with finite Lebesgue measure, i.e. $\mathcal{L}^n(\Omega) < \infty$. The classical Paley-Wiener space $PW_\Omega(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) : \text{supp}(\hat{f}) \subset \Omega\}$ is a reproducing kernel Hilbert space with kernel $Q_\Omega(\xi, \eta) = (2\pi)^{-n/2} \hat{\chi}_\Omega(\xi - \eta)$, where χ_Ω denotes the characteristic function of Ω and \hat{f}, \check{f} are the (unitary) Fourier and inverse Fourier transform of f respectively. The space $L^2(\Omega)$ is (unitarily) isomorphic to $PW_\Omega(\mathbb{R}^n)$ if we extend the elements of $L^2(\Omega)$ by zero outside $\mathbb{R}^n \setminus \Omega$. The well known Paley-Wiener states that $PW_\Omega(\mathbb{R}^n)$ is the set of restrictions of entire functions of exponential type to \mathbb{R}^n , whence these functions are real analytic and the set $PW_\Omega(\mathbb{R}^n) \cap \{\|f\| = 1\}$ is uniformly bounded, i.e. $\|f\|_\infty \leq \frac{|\Omega|}{(2\pi)^n}$. The L^p norm on \mathbb{R}^n is denoted by $\|\cdot\|_p$ where we usually omit the subscript when $p = 2$.

In the following let $\Phi : \mathbb{R}^n \rightarrow [0, \infty]$ be an inf-compact function, that is the level sets $\{\xi : \Phi(\xi) \leq t\}$ are compact (or empty) for all $t \geq 0$. We fix an orthonormal basis $\{\varphi_m\}_{m \in \mathbb{N}}$ of $PW_\Omega(\mathbb{R}^n)$ and define a “spectrum” of Φ as follows:

$$\lambda_m(\Phi) = \int_{\mathbb{R}^n} \Phi(\xi) |\varphi_m(\xi)|^2 d\xi, \quad m = 1, 2, 3, \dots \quad (1)$$

It is clear that the λ_m may be infinite and depend on the basis chosen, however, to simplify notation we do not indicate this dependence but one should keep this fact in mind when looking for minimizers. In the same spirit we associate to a basis $\{\varphi_m\}_{m \in \mathbb{N}}$ two sets of probability measures $\{\nu_N\}_{N \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ and $\{\mu_N\}_{N \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$:

$$\nu_N(f) = \frac{1}{N} \sum_{m=1}^N \int_{\mathbb{R}^n} f(\xi) |\varphi_m(\xi)|^2 d\xi, \quad N = 1, 2, 3, \dots \quad (2)$$

and

$$\mu_N(f) = \sum_{m=1}^N \int_{\mathbb{R}^n} f(\xi) |\varphi_m(N^{1/n}\xi)|^2 d\xi, \quad N = 1, 2, 3, \dots \quad (3)$$

for all $f \in C_0(\mathbb{R}^n)$ = continuous functions with compact support. It is easily seen that $S_{N,\#}\mu_N = \nu_N$ where $\#$ denotes push-forward in this context and S_N is the scaling map $(S_N f)(\xi) = f(N^{1/n}\xi)$.

Definition 1 Given Φ and $\{\varphi_m\}_{m \in \mathbb{N}}$ as above we set $M(\Lambda) = \{m \in \mathbb{N} : \lambda_m(\Phi) \leq \Lambda\}$ and denote by $\mathcal{N}(\Lambda)$ the number of elements in $M(\Lambda)$.

We have the following upper bound:

Proposition 1

$$\mathcal{N}(\Lambda) \leq \frac{|\Omega|}{(2\pi)^n} \inf_{0 < \varepsilon < 1} \frac{\mathcal{L}^n(\{\Phi \leq \frac{\Lambda}{\varepsilon}\})}{1 - \varepsilon} \quad (4)$$

For any orthonormal basis $\{\varphi_m\}_{m \in \mathbb{N}}$ of $PW_\Omega(\mathbb{R}^n)$ the kernel Q_Ω has the representation

$$Q_\Omega(\xi, \eta) = (2\pi)^{-n/2} \hat{\chi}_\Omega(\xi - \eta) = \sum_{m=1}^{\infty} \varphi_m(\xi) \bar{\varphi}_m(\eta)$$

and the trace is bounded on the whole of \mathbb{R}^n , indeed

$$\lim_{N \rightarrow \infty} \sum_{m=1}^N |\varphi_m(\xi)|^2 = \frac{|\Omega|}{(2\pi)^n} \quad (5)$$

uniformly on compact subsets (also follows from Bessel's inequality).

Definition 2 The Fermi ball associated to Ω is determined by the requirement

$$\frac{|\Omega|}{(2\pi)^n} |B_{\kappa_F}| = 1.$$

Thus the radius κ_F is given by

$$\kappa_F = \frac{2\pi}{(\omega_n |\Omega|)^{1/n}}, \quad (6)$$

where ω_n is the volume of the unit ball in \mathbb{R}^n .

Example 1 Set $\Phi_p(\xi) = |\xi|^p$, $p > 0$, then Φ is certainly inf-compact and we get

$$\mathcal{N}(\Lambda) \leq \frac{|\Omega|}{(2\pi)^n} \inf_{0 < \varepsilon < 1} \frac{\omega_n \left(\frac{\Lambda}{\varepsilon}\right)^{n/p}}{1 - \varepsilon} = \frac{|\Omega| \omega_n \Lambda^{n/p}}{(2\pi)^n} \inf_{0 < \varepsilon < 1} \frac{\varepsilon^{-n/p}}{1 - \varepsilon} = C_{n,p} \frac{|\Omega| \omega_n \Lambda^{n/p}}{(2\pi)^n}.$$

The function $\frac{\varepsilon^{-n/p}}{1 - \varepsilon}$ attains its minimum in $(0, 1)$ at $\varepsilon_0 = \frac{n}{n+p}$ so that $C_{n,p} = \frac{n+p}{p \left(\frac{n}{n+p}\right)^{n/p}}$. Now if

$N = 2\tilde{C}_n \left(\frac{\Lambda^{1/p}}{\kappa_F}\right)^n \geq 2N(\Lambda)$ then we have (at least half of the λ_m must be $> \Lambda$):

$$\sum_{j=1}^N \lambda_j(\Phi_p) \geq \frac{N}{2} \Lambda \geq \frac{N}{2} \left(\frac{N}{2\tilde{C}_n}\right)^{p/n} \kappa_F^p \sim N^{1+p/n} \kappa_F^p.$$

where $\tilde{C}_n \geq C_{n,p}$ such that N is an integer.

Definition 3 For any $N=1,2,3,\dots$ we define the quantities

$$\omega_\Phi(N) := \frac{\int_{\mathbb{R}^n} \Phi(N^{-1/n}\xi) d\nu_N(\xi)}{N \int_{\mathbb{R}^n} \Phi(\xi) d\nu_N(\xi)}.$$

For instance if Φ is homogeneous of degree p then $\omega_\Phi(N) = N^{-(1+\frac{p}{n})}$.

Proposition 2 Let Φ and $\{\varphi_m\}_{m \in \mathbb{N}}$ as above and suppose that

$$\sup_N \left[\omega_N(\Phi) \sum_{m=1}^N \lambda_m(\Phi) \right] < \infty \quad (7)$$

then the sequence of measures $\{\mu_N\}_{N \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ is tight and therefore (by Prokhorov's theorem) a subsequence μ_{N_j} converges weakly to a measure $\mu_\star \in \mathcal{P}(\mathbb{R}^n)$. Moreover we have that

$$\liminf_{j \rightarrow \infty} \int_{\mathbb{R}^n} \Phi(\xi) d\mu_{N_j}(\xi) \geq \int_{\mathbb{R}^n} \Phi(\xi) d\mu_\star(\xi).$$

It might be instructive to consider the case $\Phi = \Phi_p$.

Example 2 If $\Phi(\xi) = |\xi|^p$ then we get

$$\int_{\mathbb{R}^n} |\xi|^p d\mu_N(\xi) = N^{-(1+\frac{p}{n})} \sum_{m=1}^N \lambda_m$$

which, when setting $p = 2$ and choosing the basis $\{\varphi_m\}_m = \{\hat{u}_m\}_m$, where $\Delta u_m + \lambda_m u_m = 0$ in Ω and $u_m = 0$ in $\mathbb{R}^n \setminus \Omega$, yields by the well known Weyl law (provided that it is valid for Ω):

$$\lim_{N \rightarrow \infty} N^{-(1+\frac{2}{n})} \sum_{m=1}^N \lambda_m = \frac{n}{n+2} \kappa_F^2.$$

In a similar way we get when using Weyl's law for the fractional Laplacian [G]:

$$\lim_{N \rightarrow \infty} N^{-(1+\frac{p}{n})} \sum_{m=1}^N \lambda_m = \frac{n}{n+p} \kappa_F^p.$$

It is obvious that the right hand side above is $\int_{B_{\kappa_F}} |\xi|^p d\xi$. Since (7) is certainly satisfied one may wonder if $d\mu_\star = \chi_{B_{\kappa_F}}(\xi) d\xi$ holds.

A variant of the “bathtub” lemma yields:

Proposition 3 *Let Φ and $\{\varphi_m\}_{m \in \mathbb{N}}$ be given, then for all $N \in \mathbb{N}$:*

$$\int \Phi(\xi) d\mu_N(\xi) \geq \frac{|\Omega|}{(2\pi)^n} \int_{\{\Phi < \tau\}} \Phi(\xi) d\xi + c_0 \frac{|\Omega|\tau}{(2\pi)^n} \mathcal{L}^n(\{\xi : \Phi(\xi) = \tau\}),$$

where

$$\tau = \sup \left\{ t : \mathcal{L}^n(\{\Phi < t\}) \leq \frac{(2\pi)^n}{|\Omega|} \right\} \quad (8)$$

and if $\mathcal{L}^n(\{\xi : \Phi(\xi) = \tau\}) > 0$ then

$$c_0 = \frac{\frac{(2\pi)^n}{|\Omega|} - \mathcal{L}^n(\{\xi : \Phi(\xi) < \tau\})}{\mathcal{L}^n(\{\xi : \Phi(\xi) = \tau\})}.$$

Otherwise $c_0 = 0$.

Now it is apparent to suspect that for a “minimal” basis of $PW_\Omega(\mathbb{R}^n)$ the measures μ_N converge vaguely (at least) or even better, weakly to

$$d\mu_\Phi = \frac{|\Omega|}{(2\pi)^n} [\chi_{\{\Phi < \tau\}}(\xi) + c_0 \chi_{\{\Phi = \tau\}}(\xi)] d\xi. \quad (9)$$

For the example $\Phi = |\xi|^p$ again we get $\tau = \kappa_F^p$ and we can choose $c_0 = 0$ since $\mathcal{L}^n(\{\xi : |\xi| = \kappa_F\}) = 0$.

Corollary 1 *Let Φ and $\{\varphi_m\}_{m \in \mathbb{N}}$ be given and assume that*

$$\Lambda(\Phi) := \liminf_{N \rightarrow \infty} \omega_N(\Phi) \sum_{m=1}^N \lambda_m(\Phi) = \int_{\mathbb{R}^n} \Phi d\mu_\Phi < \infty$$

where $d\mu_\Phi = \frac{|\Omega|}{(2\pi)^n} [\chi_{\{\Phi < \tau\}}(\xi) + c_0 \chi_{\{\Phi = \tau\}}(\xi)] d\xi$, then there is a subsequence $\{N_j\}$ and a measure $\mu_\star \in \mathcal{P}(\mathbb{R}^n)$ such that $\mu_{N_j}(f) \rightarrow \mu_\star(f), \forall f \in C_b(\mathbb{R}^n)$ and

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} \Phi(\xi) d\mu_{N_j}(\xi) = \int \Phi(\xi) d\mu_\star(\xi) = \Lambda(\Phi).$$

Moreover the basis $\{\varphi_m\}_{m \in \mathbb{N}}$ is minimal in the sense that $\Lambda(\Phi)$ is the smallest value than can be achieved by the associated sequence of measures $\{\mu_N^\varphi\}_{N \geq 1}$.

Question 1 When do we have $\mu_\star = \mu_\Phi$?

Obviously one is inclined trying to prove that $d\mu_\star(\xi) = \varphi(\xi) d\xi$, for some $\varphi \in L^\infty(\mathbb{R}^n)$, and that $\|\varphi\|_\infty \leq \frac{|\Omega|}{(2\pi)^n}$. Then we had that $\varphi \in \mathcal{C}$ and consequently φ would be a minimizer. Then a condition like $\mathcal{L}^n(\{\xi : \Phi(\xi) = \tau\}) = 0$ guarantees uniqueness and we were done. However, due to the non-separability of $L^\infty(\mathbb{R}^n)$, it is by no means granted that the sequence

$$G_N(\xi) = \sum_{m=1}^N |\varphi_m(N^{1/n}\xi)|^2$$

has a (weak \star) convergent subsequence in $L^\infty(\mathbb{R}^n)$. Yet we can show:

Proposition 4 *Under the conditions of Corollary 1, the limit measure is absolutely continuous with respect to Lebesgue measure: $d\mu_\star(\xi) = G_\star(\xi)d\xi$, with $G_\star \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ and*

$$\|G_\star\|_p \leq \left(\frac{|\Omega|}{(2\pi)^n} \right)^{1-1/p}, \forall p \text{ such that } 1 \leq p < \infty.$$

Moreover we have $\|G_\star\|_1 = 1$.

Now, since we know that $G_\star \in L^\infty(\mathbb{R}^n)$ we also have that $\|G_\star\|_\infty \leq \frac{|\Omega|}{(2\pi)^n}$ by taking the limit $p \rightarrow \infty$ above. Therefore we get a partial answer to question (1) :

Corollary 2 *Under the conditions of Corollary 1, and if moreover $\mathcal{L}^n(\{\xi : \Phi(\xi) = \tau\}) = 0$, then*

$$d\mu_\star = \frac{|\Omega|}{(2\pi)^n} \chi_{\{\Phi < \tau\}}(\xi) d\xi,$$

that is $G_\star(\xi) = \frac{|\Omega|}{(2\pi)^n} \chi_{\{\Phi < \tau\}}(\xi)$.

2 Proofs

2.1 Proposition 1

By definition 1 we have for $m \in M(\Lambda)$ and $t > 0$:

$$t \int_{\{\Phi > t\}} |\varphi_m(\xi)|^2 \leq \int_{\{\Phi > t\}} \Phi(\xi) |\varphi_m(\xi)|^2 \leq \lambda_m(\Phi) = \int_{\mathbb{R}^n} \Phi(\xi) |\varphi_m(\xi)|^2 \leq \Lambda,$$

thus

$$\int_{\{\Phi \leq t\}} |\varphi_m(\xi)|^2 \geq 1 - \frac{\Lambda}{t}, \forall m \in M(\Lambda), t > 0.$$

Therefore by (5) :

$$\mathcal{N}(\Lambda) \left(1 - \frac{\Lambda}{t} \right) \leq \sum_{m \in M(\Lambda)} \int_{\{\Phi \leq t\}} |\varphi_m(\xi)|^2 d\xi \leq \frac{|\Omega|}{(2\pi)^n} \mathcal{L}^n(\{\Phi \leq t\}).$$

Set $\Lambda = \varepsilon t$, then we get

$$\mathcal{N}(\Lambda)(1 - \varepsilon) \leq \frac{|\Omega|}{(2\pi)^n} \mathcal{L}^n \left(\left\{ \Phi \leq \frac{\Lambda}{\varepsilon} \right\} \right)$$

for any $\varepsilon \in (0, 1)$ which proves (4) .

2.2 Proposition 1

By assumption $C = \sup_N \left[\omega_N(\Phi) \sum_{m=1}^N \lambda_m(\Phi) \right] < \infty$. Therefore we have for all $N \geq 1$:

$$\frac{\int_{\mathbb{R}^n} \Phi(N^{-1/n} \xi) d\nu_N(\xi)}{N \int_{\mathbb{R}^n} \Phi(\xi) d\nu_N(\xi)} \sum_{m=1}^N \lambda_m(\Phi) \leq C.$$

By the relations 1, 2 and 3 we get

$$\int_{\mathbb{R}^n} \Phi(\xi) d\mu_N(\xi) = \frac{1}{N} \int_{\mathbb{R}^n} \Phi(N^{-1/n} \xi) d\nu_N(\xi) \leq C.$$

Recall that a subset A of $\mathcal{P}(\mathbb{R}^n)$ is called *tight* if for any $\varepsilon > 0$ exists a compact subset K_ε of \mathbb{R}^n such that $\mu(K_\varepsilon) \geq 1 - \varepsilon$ for all $\mu \in A$. When we set $K_j = L_\Phi(j) = \{\xi : \Phi(\xi) \leq j\}$, $j \in \mathbb{N}$, then for any $\mu \in \{\mu_m\}_{m \in \mathbb{N}}$

$$j\mu(\mathbb{R}^n \setminus K_j) \leq \int_{\mathbb{R}^n \setminus K_j} \Phi(\xi) d\mu(\xi) \leq \int_{\mathbb{R}^n} \Phi(\xi) d\mu(\xi) \leq C,$$

thus

$$1 - \mu(K_j) \leq \frac{C}{j} \Rightarrow \mu(K_j) \geq 1 - \frac{C}{j}.$$

Since Φ is inf-compact, all the sets K_j are compact by definition, so this proves that the sequence $\{\mu_m\}_{m \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathbb{R}^n)$. Next, by Prokhorov's theorem [2], there is a subsequence μ_{m_j} which converges weakly to a measure $\mu_\star \in \mathcal{P}(\mathbb{R}^n)$. Furthermore we have by the lower semicontinuity of the function Φ (which trivially follows by the compactness of its level sets):

$$\liminf_{j \rightarrow \infty} \int \Phi(x) d\mu_{m_j}(x) \geq \int \Phi(x) d\mu_\star(x).$$

2.3 Proposition 3

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\mu(\{x : \Phi(x) > t\}) < \infty$ for all $t \in \mathbb{R}$. Then we want to show that

$$\mathcal{E}_\Phi(\varphi_0) = \inf_{\varphi \in \mathcal{C}} \int \Phi(x) \varphi(x) d\mu(x) = \int_{\{\Phi < \tau\}} \Phi(x) d\mu(x) + c_0 \tau \mu(\{x : \Phi(x) = \tau\}),$$

where

$$\mathcal{C} = \left\{ \varphi : 0 \leq \varphi(x) \leq 1, \int \varphi d\mu = A \right\} \cap \{\varphi \text{ is } \mu\text{-measurable}\}$$

and

$$\tau = \sup\{t : \mu(x : \Phi(x) < t) \leq A\}$$

and

$$c_0 \mu(\{x : \Phi(x) = \tau\}) = A - \mu(x : \Phi(x) < \tau).$$

The minimizer

$$\varphi_0(x) = \chi_{\{\Phi < \tau\}}(x) + c \chi_{\{\Phi = \tau\}}(x)$$

is unique if either $A = \mu(x : \Phi(x) < \tau)$ or $A = \mu(x : \Phi(x) \leq \tau)$.

To prove it we show that $\mathcal{E}_\Phi(\varphi_0) \leq \mathcal{E}_\Phi(\varphi)$: note that $\varphi_0 = 1$ on $\{\Phi < \tau\}$, then (using [...] as delimiters):

$$\begin{aligned} \int \Phi(x) [\varphi_0(x) - \varphi(x)] d\mu(x) = \\ \int_{\{\Phi < \tau\}} \Phi[\varphi_0 - \varphi] d\mu + \int_{\{\Phi > \tau\}} \Phi[\varphi_0 - \varphi] d\mu + \int_{\{\Phi = \tau\}} \Phi[\varphi_0 - \varphi] d\mu \leq \end{aligned}$$

$$\begin{aligned}
& \tau \int_{\{\Phi < \tau\}} [\varphi_0 - \varphi] d\mu - \int_{\{\Phi > \tau\}} \Phi \varphi d\mu + \tau \int_{\{\Phi = \tau\}} [\varphi_0 - \varphi] d\mu \\
& \leq \tau \int_{\{\Phi < \tau\}} [\varphi_0 - \varphi] d\mu - \tau \int_{\{\Phi > \tau\}} \varphi d\mu + \tau \int_{\{\Phi = \tau\}} [\varphi_0 - \varphi] d\mu \\
& = \tau \int_{\{\Phi \leq \tau\}} \varphi_0 d\mu - \tau \int \varphi d\mu = 0.
\end{aligned}$$

If $\mu(\{\Phi < \tau\}) < A$, while $\mu(\{\Phi = \tau\}) > A$, then the difference mass $A - \mu(\{\Phi < \tau\})$ has to be distributed on the level set $\{\Phi = \tau\}$, what can be done in several ways, thus the minimizer is not unique in this case.

Recalling the form of the measures μ_N , see (3), and letting $A^{-1} = \frac{|\Omega|}{(2\pi)^n}$ we notice that the functions $\psi_{m,N}(x) = A|\varphi_m(N^{1/n}\xi)|^2$ are uniformly bounded by 1 and therefore elements of the admissible set \mathcal{C} , so that

$$\frac{|\Omega|}{(2\pi)^n} \int_{\{\Phi < \tau\}} \Phi(\xi) d\xi + c_0 \frac{|\Omega|\tau}{(2\pi)^n} \mathcal{L}^n(\{\xi : \Phi(\xi) = \tau\})$$

is a lower bound to $\mu_N(\Phi)$ at any rate, as has been claimed.

2.4 Proposition 4

Let $F_N(\xi) = \sum_{m=1}^N |\varphi_m(\xi)|^2$ and $G_N(\xi) = F_N(N^{1/n}\xi)$. Then the sequences $\{F_N\}_{N \geq 1}$ and $\{G_N\}_{N \geq 1}$ are uniformly bounded on \mathbb{R}^n by $\frac{|\Omega|}{(2\pi)^n}$. Now we have

$$\int_{\mathbb{R}^n} F_N(N^{1/n}\xi)^p d\xi = \frac{1}{N} \int_{\mathbb{R}^n} F_N(\xi)^p d\xi = \frac{1}{N} \int_{\mathbb{R}^n} F_N(\xi) F_N(\xi)^{p-1} d\xi \leq \left(\frac{|\Omega|}{(2\pi)^n} \right)^{p-1},$$

thus $G_N \in L^p(\mathbb{R}^n)$ for all $p \geq 1$, and $\|G_N\|_p \leq \left(\frac{|\Omega|}{(2\pi)^n} \right)^{1-1/p}$. Now, denote by G_{N_m} the subsequence which converges to the limit measure μ_\star , that is

$$\lim_{m \rightarrow \infty} \int \varphi(\xi) G_{N_m}(\xi) d\xi = \int \varphi(\xi) d\mu_\star(\xi) = \mu_\star(\varphi), \forall \varphi \in C_b(\mathbb{R}^n),$$

then

$$\lim_{m \rightarrow \infty} \int G_{N_m}(\xi) d\xi = \lim_{m \rightarrow \infty} \|G_{N_m}\|_1 = \mu_\star(1) = 1.$$

The set $S := \{G_N : N \in \mathbb{N}\} \subset L^1(\mathbb{R}^n)$ is uniformly integrable. Indeed,

$$\int_A G_N(\xi) d\xi = \int_A F_N(N^{1/n}\xi) d\xi = \frac{1}{N} \int_{N^{1/n}A} F_N(\eta) d\eta \leq \frac{|\Omega|}{(2\pi)^n} |A|.$$

Hence, by the Dunford-Pettis theorem, S is relatively weakly compact in $L^1(\mathbb{R}^n)$ and thus the Eberlein-Smulian theorem guarantees the relatively weak sequential compactness of S . So, there is another subsequence, $G_{N_{m_j}}$ which converges weakly in $L^1(\mathbb{R}^n)$ to a function G_\star , that is

$$\lim_{j \rightarrow \infty} \int G_{N_{m_j}}(\xi) f(\xi) d\xi = \int f(\xi) G_\star(\xi) d\xi, \forall f \in L^\infty(\mathbb{R}^n).$$

Since $C_b(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ (although not dense) we must have $d\mu_\star(\xi) = G_\star(\xi)d\xi$, and consequently

$$\|G_\star\|_1 = 1.$$

Clearly, we also have that

$$G_{N_{m_j}} \xrightarrow{\text{weak}-L^p} G_\star$$

for all $1 < p < \infty$ and therefore

$$\|G_\star\|_p \leq \liminf_{j \rightarrow \infty} \|G_{N_{m_j}}\|_p \leq \left(\frac{|\Omega|}{(2\pi)^n} \right)^{1-1/p}, \forall 1 \leq p < \infty.$$

It remains the case $p = \infty$. Suppose $G_\star \notin L^\infty(\mathbb{R}^n)$ then we can find a constant $C > \frac{|\Omega|}{(2\pi)^n}$ and a measurable set $A \subset \mathbb{R}^n$ satisfying $\infty > \mathcal{L}^n(A) > 0$ such that $|G_\star(\xi)| \geq C$ for all $\xi \in A$. However, this implies

$$\liminf_{p \rightarrow \infty} \|G_\star\|_p \geq C > \frac{|\Omega|}{(2\pi)^n},$$

which contradicts $\|G_\star\|_p \leq \left(\frac{|\Omega|}{(2\pi)^n} \right)^{1-1/p}$.

3 Remarks and Examples

This section provides some examples and thoughts which motivated the investigation reported in the first section.

3.1 Dirichlet Laplacian

Let Ω be an open subset of \mathbb{R}^n with compact closure $\bar{\Omega}$ and denote by λ_j , $u_j(x)$, $j \in \mathbb{N}$ the eigenvalues and eigenfunctions to the Dirichlet problem:

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0, & x \in \Omega \\ u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (10)$$

We assume that the eigenfunctions are orthogonal, normalized and ordered by increasing eigenvalues, thus

$$\|u_j\| = 1, \quad \|\nabla u_j\| = \sqrt{\lambda_j},$$

where $\|f\| = \sqrt{\langle f, f \rangle}$ denotes the L^2 norm on \mathbb{R}^n . We use the Fourier transform in the unitary form

$$\hat{u}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-i\langle k, x \rangle} dx.$$

Bessel's inequality immediatley shows that the Fourier transforms of the eigenfunctions to (10) are uniformly bounded on \mathbb{R}^n and moreover:

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N |\hat{u}_j(k)|^2 = \frac{|\Omega|}{(2\pi)^n}$$

for all $k \in \mathbb{R}^n$, where the convergence is uniform on compact sets. Since the eigenfunctions are certainly in $L^1(\mathbb{R}^n)$ we get by the Riemann-Lebesgue lemma that the \hat{u}_j are continuous and

$$\lim_{|k| \rightarrow \infty} |\hat{u}_j(k)| = 0.$$

Therefore the convergence of the partial sums above cannot be uniform on the whole of \mathbb{R}^n . Furthermore, we have by Parseval's theorem

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |\hat{u}_j(k)|^2 dk = N$$

and

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |k|^2 |\hat{u}_j(k)|^2 dk = \sum_{j=1}^N \int_{\mathbb{R}^n} |\nabla u_j(x)|^2 dx = \sum_{j=1}^N \lambda_j,$$

so that both expressions above tend to infinity if N does. If we scale each sum by the transformation $k \rightarrow a_N k$, $a_N > 0$, we get the equations:

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |\hat{u}_j(a_N k)|^2 dk = \frac{N}{a_N^n}$$

and

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |k|^2 |\hat{u}_j(a_N k)|^2 dk = \frac{1}{a_N^{n+2}} \sum_{j=1}^N \lambda_j,$$

yielding - when eliminating the a_N on the right hand sides:

$$\frac{\sum_{j=1}^N \int_{\mathbb{R}^n} |k|^2 |\hat{u}_j(a_N k)|^2 dk}{\left(\sum_{j=1}^N \int_{\mathbb{R}^n} |\hat{u}_j(a_N k)|^2 dk \right)^{1+2/n}} = N^{1-\frac{2}{n}} \sum_{j=1}^N \lambda_j.$$

The most obvious choice seems to be $a_j = N^{1/n}$ for all j . Indeed, the nominator becomes unity and the last equation simplifies to:

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |k|^2 |\hat{u}_j(N^{1/n} k)|^2 dk = N^{1-\frac{2}{n}} \sum_{j=1}^N \lambda_j. \quad (11)$$

We also notice by Fatou's Lemma

$$\int_{\mathbb{R}^n} \liminf_{N \rightarrow \infty} \sum_{j=1}^N |\hat{u}_j(a_N k)|^2 dk \leq \liminf_{N \rightarrow \infty} \frac{N}{a_N^n},$$

that means if we scale too strong then the partial sums may converge to zero a.e. whenever

$$\liminf_{N \rightarrow \infty} \frac{N}{a_N^n} = 0.$$

However, $\lim_{N \rightarrow \infty} \sum_{j=1}^N |\hat{u}_j(0)|^2 = \frac{|\Omega|}{(2\pi)^n}$ exists in any case.

Question 2 Does the limit

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N |\hat{u}_j(N^{1/n}k)|^2$$

exists pointwise?

Apparently, the same procedure can also be repeated for other operators instead of the Laplacian. If we take the fractional Laplacian ($p > 0$) instead we obtain

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |k|^p |\hat{u}_j(N^{1/n}k)|^2 dk = N^{1-\frac{p}{n}} \sum_{j=1}^N \lambda_j, \quad (12)$$

where the meaning of the involved quantities has to be changed correspondingly, of course. However, the question remains unchanged, indicating a general phenomenon. It will be instructive to have a look at the case $n = 1$.

3.1.1 Case $n = 1$

Let $\bar{\Omega} = [0, \pi]$. Then the eigenfunctions to (1) have the simple form

$$u_m(x) = \sqrt{\frac{2}{\pi}} \chi_{[0, \pi]}(x) \sin(mx)$$

with corresponding eigenvalues $\lambda_m = m^2$. The Fourier transforms are

$$\hat{u}_m(k) = \frac{1}{\pi} \int_0^\pi \sin(mx) e^{-ikx} dx.$$

Now let us calculate

$$F_N(k) = \sum_{m=1}^N |\hat{u}_m(k)|^2.$$

Instead of using the Fourier transforms \hat{u}_m above directly we will use the following formula which is easily verified for the general case $\Omega \subset \mathbb{R}^n$:

$$F_N(k) = \frac{1}{(2\pi)^n} \sum_{m=1}^N \frac{|\int_{\Omega} \operatorname{div}(e^{-i\langle k, x \rangle} u_m(x)) dx|^2}{(|k|^2 - \lambda_m^2)}.$$

If Ω is sufficiently smooth then the numerators above are boundary integrals which in our case $\Omega = [0, \pi]$ are easily read off, yielding ($\lambda_m = m^2$):

$$F_N(k) = \frac{2}{\pi^2} \sum_{m=1}^N \frac{m^2(1 - (-1)^m \cos(\pi k))}{(k^2 - m^2)^2}.$$

The figure shows the graphs of $F_N(Nk)$ for the values $N = 1, 5, 50, 500$. Thus one might ask

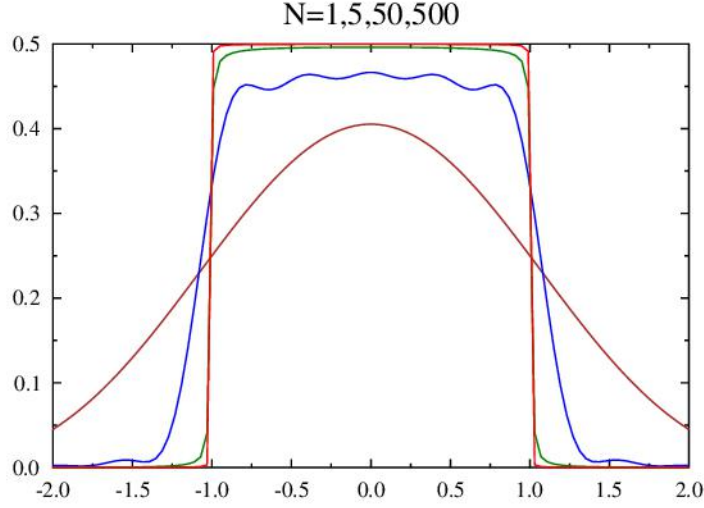


Figure 1:

Question 3 The diagram for $F_N(Nk)$, $N \in [1, 5, 50, 500]$ suggests that

$$\lim_{N \rightarrow \infty} F_N(Nk) = \begin{cases} \frac{1}{2} \dots \text{for } |k| < 1 \\ 0 \dots \text{for } |k| > 1 \end{cases}$$

Does this hold for general dimensions and domains when suitably scaled?

Indeed, for $|k| > 1$, we have

$$\begin{aligned} F_N(Nk) &= \frac{2}{\pi^2} \sum_{m=1}^N \frac{m^2(1 - (-1)^m \cos(\pi Nk))}{(N^2k^2 - m^2)^2} \leq \frac{2}{\pi^2} \sum_{m=1}^N \frac{2m^2}{N^4 \left(k^2 - \frac{m^2}{N^2}\right)^2} \leq \frac{4}{N^4 \pi^2 (k^2 - 1)^2} \sum_{m=1}^N m^2 \\ &= \frac{4N(2N^2 + 3N + 1)}{N^4 \pi^2 (k^2 - 1)^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

For $k = 0$ we get

$$F_N(N \cdot 0) = \frac{2}{\pi^2} \sum_{m=1}^N \frac{1 - (-1)^m}{m^2},$$

therefore

$$\lim_{N \rightarrow \infty} F_N(0) = F_\infty(0) = \frac{2}{\pi^2} \left(\sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \right) = \frac{2}{\pi^2} \left(\frac{\pi^2}{6} + \frac{\pi^2}{12} \right) = \frac{1}{2}$$

as expected $\left(\frac{|\Omega|}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}\right)$. Since $\int_{[-1,1]} F_N(Nk) dk = 1$ and $0 \leq F \leq \frac{1}{2} \Rightarrow$ actually $F_N(Nk) = \frac{1}{2}$ for $|k| < 1$, at least almost everywhere.

To conclude the example we remark that for the cube $[0, \pi]^n \subset \mathbb{R}^n$ we get the same conclusion when we consider the expression

$$F_N(N^{1/n}k) = \left(\frac{2}{\pi^2}\right)^n \sum_{m_1=1}^M \cdots \sum_{m_n=1}^M \prod_{j=1}^n \frac{m_j^2(1 - (-1)^{m_j} \cos(\pi M k))}{(M^2 k^2 - m_j^2)^2},$$

where $N = M^n$.

Remark 1 Repeating the calculations for $\left\{e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n \in \mathbb{Z}}$ on $[0, 2\pi]$ we get for the Fourier transforms and the radius:

$$|\hat{e}_n(k)|^2 = \frac{1}{2\pi^2} \frac{1 - \cos(2\pi k)}{(n - k)^2}, \quad |\Omega| = 2\pi \Rightarrow \frac{|\Omega|}{(2\pi)} = 1 \Rightarrow \kappa_F(\Omega) = 1.$$

Consequently

$$F_N(Nk) = \sum_{n=-N}^N |\hat{e}_n(Nk)|^2 = \sum_{n=-N}^N \frac{1}{2\pi^2} \frac{2 \sin^2(N\pi k)}{(n - Nk)^2} = \frac{1}{\pi^2} \sum_{n=-N}^N \frac{\sin^2(N\pi k)}{(n - Nk)^2},$$

and

$$\sum_{n=-N}^N |\hat{e}_n(Nk)|^2 \rightarrow \theta(1 - |k|),$$

which shows that the boundary conditions are not really relevant for the convergence.

Taking the Haar system, $x \in [0, 1]$, $f_0(x) = 1$.

$$f_{j,n}(x) = 2^{\frac{n-1}{2}} \chi_{[\frac{2j-2}{2^n}, \frac{2j-1}{2^n}]}(x) - 2^{\frac{n-1}{2}} \chi_{[\frac{2j-1}{2^n}, \frac{2j}{2^n}]}(x)$$

$j = 1 \dots 2^{n-1}$, $n = 1, 2, 3, \dots$, we obtain

$$|\hat{f}_{j,n}(k)|^2 = 2^{n+2} \frac{\sin^4\left(\frac{k}{2^{n+1}}\right)}{\pi k^2},$$

which shows that not every Fourier image of a basis in $L^2(\Omega)$, although in $PW_\Omega(\mathbb{R})$, gives raise to the observed limit behaviour.

3.2 Pointwise Convergence and Scheffe's Theorem

Functions in $PW_\Omega(\mathbb{R}^n)$ are uniformly bounded and vanish at infinity. The following simple thoughts show that this can be sufficient under certain conditions to get convergence to a Heaviside function.

Let $f_n : \mathbb{R}^d \rightarrow [0, 1]$ be a sequence of functions such that

- a) $\lim_{n \rightarrow \infty} f_n(x) = 1$.
- b) $\lim_{|x| \rightarrow \infty} f_n(x) = 0$.

Let $\{a_n\} \subset \mathbb{R}$ be a null sequence. From (b) we conclude that there exists a sequence $\{r_n\}$ such that

$$f_n(x) \leq a_n, \forall |x| \geq r_n.$$

Therefore we have

$$f_n(\lambda_n x) \leq a_n, \forall |x| \geq \frac{r_n}{\lambda_n}.$$

Thus, if for another sequence $\{\lambda_n\}$ exists

$$\limsup_{n \rightarrow \infty} \frac{r_n}{\lambda_n} = x^*$$

then

$$\lim_{n \rightarrow \infty} f_n(\lambda_n x) = 0, \forall |x| > x^*.$$

On the other hand, suppose

$$f_n(x) \geq 1 - a_n, \forall |x| \leq \rho_n$$

then along the same lines:

$$f_n(\mu_n x) \geq 1 - a_n, \forall |x| \leq \frac{\rho_n}{\mu_n}.$$

$$\liminf_{n \rightarrow \infty} \frac{\rho_n}{\mu_n} = x_* \Rightarrow \lim_{n \rightarrow \infty} f_n(\mu_n x) = 1, \forall |x| < x_*.$$

Consequently

$$\frac{r_n}{\lambda_n} \sim \frac{\rho_n}{\mu_n} \Rightarrow x_* = x^* \quad i.e. \quad \lim_{n \rightarrow \infty} \frac{r_n}{\lambda_n} = \lim_{n \rightarrow \infty} \frac{\rho_n}{\mu_n}$$

That means, if we can show pointwise convergence then by Scheffe's theorem [3] we also have convergence in measure (more precisely: the associated densities).

The simple example $f_n(x) = 1 / \left(1 + \frac{x^4}{n^4}\right)$, $d = 1$, however, gives

$$\lim_{n \rightarrow \infty} f_n\left(\frac{\pi n}{\sqrt{2}} x\right) = \frac{1}{1 + \frac{\pi^4 x^4}{4}},$$

in spite of $\int x^2 f_n(x) \sim n^3$ is finite for all $n \in \mathbb{N}$. Although the f_n are not in $PW_\Omega(\mathbb{R})$, they are real analytic and have finite kinetic energy. Apparently, a universal condition granting convergence to a “Fermi ball” cannot be stated easily.

3.3 Weyl's Law and the Fermi Ball

For the Dirichlet Laplacian (10) the asymptotics of the eigenvalues λ_m as $m \rightarrow \infty$ is well known

$$\lambda_m = c_n(\Omega) m^{2/n} + o(1) \tag{13}$$

and so for the sum of the first N eigenvalues

$$\sum_{m=1}^N \lambda_m = c_n(\Omega) \frac{n}{n+2} N^{1+\frac{2}{n}} + o(1)$$

where

$$c_n(\Omega) = \frac{4\pi\Gamma\left(1 + \frac{n}{2}\right)^{2/n}}{|\Omega|^{2/n}} := \kappa_F^2.$$

When we recall (11) we see that the limit exists:

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N \int_{\mathbb{R}^n} |k|^2 |\hat{u}_j(N^{1/n}k)|^2 dk = N^{1-\frac{2}{n}} \sum_{j=1}^N \lambda_j = \frac{n}{n+2} \kappa_F^2.$$

The quantity $\kappa_F = \kappa_F(\Omega)$ is sometimes called the Fermi radius, i.e. more precisely the Fermi momentum ($\hbar = 1$). It is the radius of the ball B_{κ_F} such that

$$\frac{|\Omega|}{(2\pi)^n} |B_{\kappa_F}| = 1. \quad (14)$$

Indeed, if we recall that the volume of the unit ball is given by

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}$$

then

$$\frac{|\Omega|}{(2\pi)^n} \omega_n \kappa_F^n = 1 \Rightarrow \kappa_F = \frac{2\sqrt{\pi}\Gamma(1 + n/2)^{1/n}}{|\Omega|^{1/n}} = \sqrt{c_n(\Omega)} = \frac{2\pi}{(\omega_n |\Omega|)^{1/n}}.$$

When we look at equation (12) then we might suspect that for the fractional Laplacian (symbol $|k|^p$)

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N \int_{\mathbb{R}^n} |k|^p |\hat{u}_j(N^{1/n}k)|^2 dk = N^{1-\frac{p}{n}} \sum_{j=1}^N \lambda_j = \frac{n}{n+p} \kappa_F^p$$

holds analogously:

$$\sum_{j=1}^N |\hat{u}_j(N^{1/n}k)|^2 \xrightarrow{N \rightarrow \infty} \frac{|\Omega|}{(2\pi)^n} \chi_{B_{\kappa_F}(k)}.$$

Actually, if we assume that

$$\sum_{j=1}^N \int_{\mathbb{R}^n} |k|^p |\hat{u}_j(N^{1/n}k)|^2 dk \xrightarrow{N \rightarrow \infty} \frac{|\Omega|}{(2\pi)^n} \int |k|^p \chi_{B_{\kappa_F}(k)} dk$$

then the last integral gives the correct asymptotic value.

3.4 Mean Values

Definition 4 Let $u \in C(\mathbb{R}^n)$. The spherical average $M_u(x, r)$ over a sphere with radius r and center x is defined as:

$$M_u(x, r) = \frac{1}{n\omega_n} \int_{S^{n-1}} u(x + r\theta) d\sigma(\theta),$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})}$ denotes the volume of the unit ball in \mathbb{R}^n .

Let $\Delta u + \lambda u = 0$ in Ω . Then it is shown in [1]

$$M_u(x, r) = u(x) \frac{\Gamma\left(\frac{n}{2}\right) J_{\frac{n-2}{2}}(\sqrt{\lambda}r)}{\left(\frac{r\sqrt{\lambda}}{2}\right)^{\frac{n-2}{2}}} = u(x) P_n(\sqrt{\lambda}r). \quad (15)$$

for any sphere $\partial B_r(x) \subset \Omega$. We write P_n as

$$P_n(\xi) = \frac{(2\pi)^{n/2} J_{\frac{n-2}{2}}(\xi)}{n\omega_n \xi^{\frac{n-2}{2}}}.$$

It hold for example:

$$P_2(\xi) = J_0(\xi)$$

and

$$P_3(\xi) = \frac{\sin(\xi)}{\xi}.$$

If n is odd then P_n may be expressed by derivatives of P_3 :

$$P_{2m+1}(\xi) = \frac{(-1)^{m-1} 2^{2m-1} \Gamma\left(m + \frac{1}{2}\right)}{\sqrt{\pi}} \frac{d^{m-1}}{d(\xi^2)^{m-1}} \left(\frac{\sin(\xi)}{\xi} \right), \quad m = 1, 2, \dots$$

Moreover, let $\mu = \frac{n-2}{2}$, then

$$\xi^\mu P_{2\mu+2}(\xi) \sim J_\mu(\xi),$$

using Bessel's differential equation:

$$\xi^2 (\xi^\mu P_{2\mu+2}(\xi))'' + \xi (\xi^\mu P_{2\mu+2}(\xi))' + (\xi^2 - \mu^2) \xi^\mu P_{2\mu+2}(\xi) = 0$$

thus ($\xi \neq 0$)

$$P_{2\mu+2}(\xi)'' + \frac{2\mu+1}{\xi} P_{2\mu+2}(\xi)' + P_{2\mu+2}(\xi) = 0 \iff \Delta_n P_n(|x|) + P_n(|x|) = 0.$$

3.4.1 Radial Fourier Transform

Since $u_k(x) = e^{-ikx}$ is a solution to $\Delta u(x) + |k|^2 u(x) = 0$ we must have by (15) :

$$M_{u_k}(x, r) = u_k(x) P_n(|k|r) = e^{-ikx} P_n(|k|r).$$

Therefore (if $f(x) = f(|x|)$) :

$$\hat{f}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle k, x \rangle} dx = \frac{n\omega_n}{(2\pi)^{n/2}} \int_0^\infty f(r) M_{u_k}(0, r) r^{n-1} dr.$$

so that again

$$\hat{f}(\kappa) = \frac{n\omega_n}{(2\pi)^{n/2}} \int_0^\infty f(r) P_n(\kappa r) r^{n-1} dr. \quad (16)$$

This may also be written as

$$\hat{f}(k) = \hat{f}(|k|) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(|x|) P_n(|k||x|) dx. \quad (17)$$

It is obvious that the inverse transform has the same form.

3.4.2 FT of the Characteristic Function of a Ball

Let $f(r) = \begin{cases} a & \text{for } r \leq \lambda \\ 0 & \text{else.} \end{cases}$, then

$$\begin{aligned} \hat{f}(\kappa) &= \frac{an\omega_n}{(2\pi)^{n/2}} \int_0^\lambda P_n(\kappa r) r^{n-1} dr = \frac{an\omega_n}{\kappa^n (2\pi)^{n/2}} \int_0^{\kappa\lambda} P_n(\rho) \rho^{n-1} d\rho. \\ &= \frac{a}{\kappa^n (2\pi)^{n/2}} \int_{|x| < \kappa\lambda} P_n(|x|) dx = \frac{-an\omega_n(\kappa\lambda)^{n-1}}{\kappa^n (2\pi)^{n/2}} \left(\frac{d}{ds} P_n(s) \right) \Big|_{s=\kappa\lambda} \end{aligned}$$

where we have used that $\Delta P_n + P_n = 0$.

$$\frac{d}{ds} P_n(s) = -\frac{(2\pi)^{n/2}}{n\omega_n} \frac{J_{\frac{n}{2}}(s)}{s^{\frac{n-2}{2}}}$$

thus

$$\hat{\chi}_{B_\lambda}(\kappa) = \left(\frac{\lambda}{\kappa} \right)^{n/2} J_{\frac{n}{2}}(\kappa\lambda) \quad (18)$$

3.5 Kernels and Autocorrelation

To each eigenfunction u_m of (10) we associate the function

$$U_m(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u_m(x+y) \bar{u}_m(y) dy$$

which has its support in $\Omega \oplus (-\Omega)$. The Fourier transform of these functions are given by

$$\hat{U}_m(k) = |\hat{u}_m(k)|^2, m \in \mathbb{N},$$

so that we can write

$$F_N(N^{1/n}k) = \sum_{m=1}^N |\hat{u}_m(N^{1/n}k)|^2 = \sum_{m=1}^N \hat{U}_m(N^{1/n}k).$$

When we actually perform the Fourier transform of U_m we obtain

$$F_N(N^{1/n}k) = \frac{1}{N(2\pi)^{n/2}} \sum_{j=1}^N \int_{\mathbb{R}^n} U_j \left(\frac{\xi}{N^{1/n}} \right) e^{-i\langle k, \xi \rangle} d\xi. \quad (19)$$

Now let $Q_N(x, y)$ denote the kernel

$$Q_N(x, y) = \frac{1}{N} \sum_{m=1}^N u_m \left(x + \frac{y}{2N^{1/n}} \right) \bar{u}_m \left(x - \frac{y}{2N^{1/n}} \right).$$

B. Schmidt has proven in [4]

$$Q_N \xrightarrow{N \rightarrow \infty} Q_\star = (2\pi)^{-n} \chi_\Omega \otimes \hat{\chi}_{B_{\kappa_F}} \quad (20)$$

in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ if Ω is such that Weyl's Law is valid. Actually it is shown

$$Q_\star(x, y) = \chi_\Omega(x) \frac{2^{n/2} \Gamma(1 + \frac{n}{2}) J_{\frac{n}{2}}(\kappa_F |y|)}{|\Omega|(\kappa_F |y|)^{n/2}}.$$

The equivalence to (20) follows from (18) and (14).

Setting

$$\begin{aligned} G_N(x) &:= \frac{1}{N} \sum_{j=1}^N U_j\left(\frac{x}{N^{1/n}}\right) = \frac{1}{N(2\pi)^{n/2}} \sum_{j=1}^N \int_{\Omega} u_j\left(\frac{x}{N^{1/n}} + y\right) u_j(y) dy \\ &= \frac{1}{N(2\pi)^{n/2}} \sum_{j=1}^N \int_{\Omega + \frac{x}{N^{1/n}}} u_j\left(\xi + \frac{x}{2N^{1/n}}\right) u_j\left(\xi - \frac{x}{2N^{1/n}}\right) d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\Omega + \frac{x}{N^{1/n}}} Q_N(\xi, x) d\xi \end{aligned}$$

we can see that when comparing to (19):

$$\hat{G}_N(k) = F_N(N^{1/n}k)$$

and consequently by (20) and Parseval's theorem we get as $N \rightarrow \infty$:

$$G_N(x) \xrightarrow{L^2} \frac{\hat{\chi}_{B_{|k_F|(x)}}}{(2\pi)^{n/2}} |\Omega| \iff F_N(N^{1/n}k) \xrightarrow{L^2} \frac{|\Omega|}{(2\pi)^n} \chi_{B_{\kappa_F}(k)}. \quad (21)$$

In summary we can state that for the example problem (10) the scaled partial sums of $|\hat{u}_m|^2$ convergence strongly in L^2 to the “Fermi ball” $\frac{|\Omega|}{(2\pi)^n} \chi_{B_{\kappa_F}(k)}$. This is far more than the expected convergence in measure (vaguely as well as weakly). It seems that the methods of [4] may succeed for other symbols as well.

3.5.1 Rate of decay

If we assume (21) then

$$\lim_{N \rightarrow \infty} \left\| \hat{G}_N - \frac{|\Omega|}{(2\pi)^n} \chi_{B_{\kappa_F}} \right\|_{L^2(\mathbb{R}^n)} \rightarrow 0.$$

This means

$$I_N = \int_{\mathbb{R}^n} |F_N(N^{1/n}k) - \frac{|\Omega|}{(2\pi)^n} \chi_{B_{\kappa_F}}(k)|^2 dk \xrightarrow{N \rightarrow \infty} 0.$$

When expanding while using (14) we get

$$I_N = \frac{1}{N} \int_{\mathbb{R}^n} F_N(k)^2 dk - 2 \frac{|\Omega|}{(2\pi)^n} \int_{|k| \leq \kappa_F} F_N(N^{1/n}k) dk + \frac{|\Omega|}{(2\pi)^n}.$$

Recall that the $F_N(k)$ are uniformly bounded by $\frac{|\Omega|}{(2\pi)^n}$ (that's why $F_N \in L^1 \cap L^2$). Thus

$$I_N \leq 2 \frac{|\Omega|}{(2\pi)^n} \left[1 - \int_{|k| \leq \kappa_F} F_N(N^{1/n}k) dk \right].$$

Whence we see that the tricky bit for a direct prove is to show that

$$\liminf_{N \rightarrow \infty} \int_{|k| \leq \kappa_F} F_N(N^{1/n} k) dk = 1. \quad (22)$$

Because

$$\int_{\mathbb{R}^n} F_N(N^{1/n} k) dk = 1,$$

(22) means that the decay of F_N must be strong enough (but not too strong) to concentrate uniformly in B_{κ_F} . The well known concentration compactness lemma reveals that we need “tightness” so that the “momentum” cannot run away to infinity.

3.6 Polya’s Conjecture

To conclude this introductory setion we have a look at the connection of the sequence $\{F_N\}_{N \in \mathbb{N}}$ to the famous conjecture of Polya: is (13) a lower bound to the eigenvalues $\lambda_m(\Omega)$ of (10)? That is, does

$$\lambda_m(\Omega) \geq m^{2/n} \kappa_F^2$$

hold for all $m \in \mathbb{N}$? Polya himself proved this for the case when Ω is a “tiling” domain. This problem is (to our knowledge) still unsolved and even the case of the disk in \mathbb{R}^2 has not been settled yet (here too, as far as we know). An alternative formulation is, for example, using (14):

$$\frac{|\Omega|}{(2\pi)^n} |B_{\sqrt{\lambda_m}}| \geq m?$$

which, if true, would imply

$$|B_{\sqrt{\lambda_m}}| \geq m |B_{\kappa_F}|.$$

Now remembering the fact that

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N |\hat{u}_j(k)|^2 = \frac{|\Omega|}{(2\pi)^n},$$

the following equality for each m is easily established when integrating over the ball $B_{\sqrt{\lambda_m}}$:

$$\frac{|\Omega| |B_{\sqrt{\lambda_m}}|}{(2\pi)^n} - \sum_{j=1}^m \|\hat{u}_j\|^2 = \sum_{j=m+1}^{\infty} \int_{B_{\sqrt{\lambda_m}}} |\hat{u}_j(k)|^2 - \sum_{j=1}^m \int_{\mathbb{R}^n \setminus B_{\sqrt{\lambda_m}}} |\hat{u}_j(k)|^2.$$

Thus if we could show that

$$\sum_{j=1}^m \int_{\mathbb{R}^n \setminus B_{\sqrt{\lambda_m}}} |\hat{u}_j(k)|^2 \leq \sum_{j=m+1}^{\infty} \int_{B_{\sqrt{\lambda_m}}} |\hat{u}_j(k)|^2$$

holds for all $m \in \mathbb{N}$, then Polya was right. Using the information of the kintetic terms we get analogously

$$\frac{|\Omega| |B_{\sqrt{\lambda_m}}|}{(2\pi)^n} \frac{\lambda_m}{n+2} - \sum_{j=1}^m \lambda_j \|\hat{u}_j\|^2 = \sum_{j=m+1}^{\infty} \int_{B_{\sqrt{\lambda_m}}} |k|^2 |\hat{u}_j(k)|^2 - \sum_{j=1}^m \int_{\mathbb{R}^n \setminus B_{\sqrt{\lambda_m}}} |k|^2 |\hat{u}_j(k)|^2.$$

It is not to overlook that this represents quite a subtle balance problem that needs far more precise growth information of the \hat{u}_j than those which are presently known (see e.g. [6] and ref. therein).

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V 1.0, kp@scios.ch